J423. Proposed by Titu Andreescu, University of Texas at Dallas, USA

(**a**) Prove that for any real numbers a, b, c

$$a^2 + (2 - \sqrt{2})b^2 + c^2 \ge \sqrt{2}(ab - bc + ca);$$

(**b**) Find the best constant k such that for all real numbers a, b, c

$$a^2 + kb^2 + c^2 \ge \sqrt{2}(ab + bc + ca).$$

Solution by Arkady Alt, San Jose, California, USA.

(**a**) $a^2 + (2 - \sqrt{2})b^2 + c^2 - \sqrt{2}(ab - bc + ca) = a^2 - \sqrt{2}a(b + c) + (2 - \sqrt{2})b^2 + c^2 + \sqrt{2}bc =$

$$\left(a - \frac{b+c}{\sqrt{2}}\right)^2 + (2 - \sqrt{2})b^2 + c^2 + \sqrt{2}bc - \frac{(b+c)^2}{2} =$$

$$\left(a - \frac{b+c}{\sqrt{2}}\right)^2 + \frac{\left(3 - 2\sqrt{2}\right)b^2 + 2\left(\sqrt{2} - 1\right)bc + c^2}{2} =$$

$$\left(a - \frac{b+c}{\sqrt{2}}\right)^2 + \frac{\left(\left(\sqrt{2} - 1\right)b + c\right)^2}{2} \ge 0.$$

(**b**) Note that $a^2 + kb^2 + c^2 - \sqrt{2}(ab + bc + ca) = a^2 - \sqrt{2}a(b + c) + kb^2 + c^2 - \sqrt{2}bc =$

$$\left(a - \frac{b+c}{\sqrt{2}}\right)^2 + kb^2 + c^2 - \sqrt{2}bc - \frac{(b+c)^2}{2} =$$

$$\left(a - \frac{b+c}{\sqrt{2}}\right)^2 + \frac{(2k-1)b^2 + c^2 - 2\left(\sqrt{2} + 1\right)bc}{2}.$$

Assume that inequality $a^2 + kb^2 + c^2 \ge \sqrt{2}(ab + bc + ca)$. holds for any real a, b, c.

Then $\left(a - \frac{b+c}{\sqrt{2}}\right)^2 + \frac{(2k-1)b^2 + c^2 - 2\left(\sqrt{2} + 1\right)bc}{2} \ge 0$ for any real a, b, c and,

therefore, in particular, for any real b, c and $a = \frac{b+c}{\sqrt{2}}$ holds inequality

(**1**) $c^2 - 2\left(\sqrt{2} + 1\right)bc + (2k-1)b^2 \ge 0 \iff \left(c - \left(\sqrt{2} + 1\right)b\right)^2 + 2b^2\left(k - \sqrt{2} - 2\right) \ge 0$

In particular for $b = 1$ and $c = \sqrt{2} + 1$ we obtain $2\left(k - \sqrt{2} - 2\right) \ge 0 \iff k \ge \sqrt{2} + 2.$

Let now $k \ge \sqrt{2} + 2$. Then $a^2 + kb^2 + c^2 - \sqrt{2}(ab + bc + ca) =$

$$\left(a - \frac{b+c}{\sqrt{2}}\right)^2 + \frac{(2k-1)b^2 + c^2 - 2\left(\sqrt{2} + 1\right)bc}{2} =$$

$$\left(a - \frac{b+c}{\sqrt{2}}\right)^2 + \frac{\left(c - \left(\sqrt{2} + 1\right)b\right)^2 + 2b^2\left(k - \sqrt{2} - 2\right)}{2} \ge 0$$ for any real a, b, c.

Thus, $k = \sqrt{2} + 2$ is the best (minimal) constant k such that for all real numbers a, b, c inequality $a^2 + kb^2 + c^2 \ge \sqrt{2}(ab + bc + ca)$ holds.